Counting Sudoku Variants

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Sudoku

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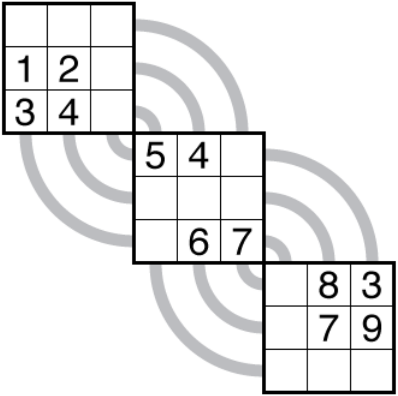
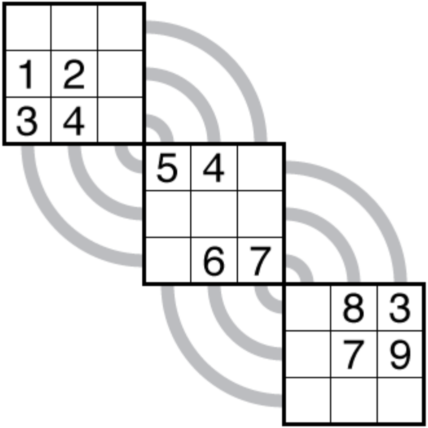
- Smallest number of clues needed for unique solution is 17 (computer 2012).

Sudo-Kurve



Example of Sudo-Kurve (from gmpuzzles.com)

Solving the Sudo-Kurve



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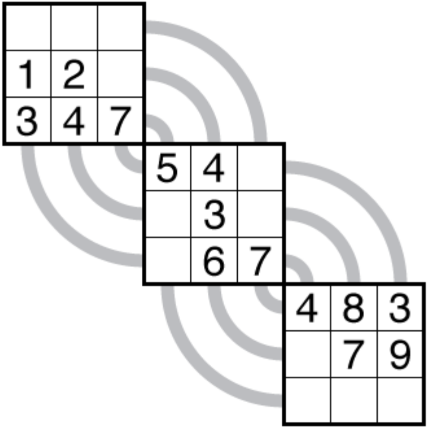

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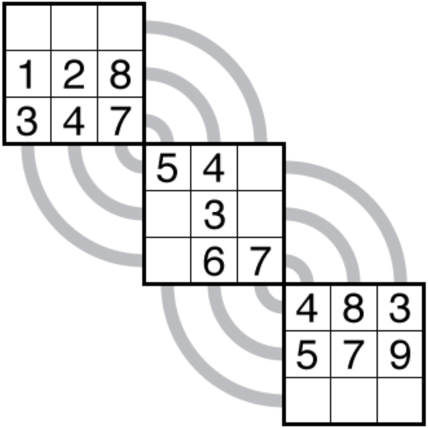

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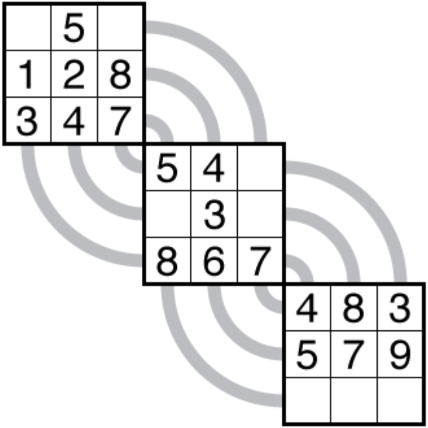

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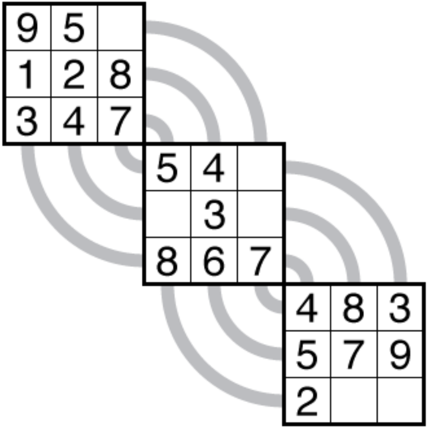

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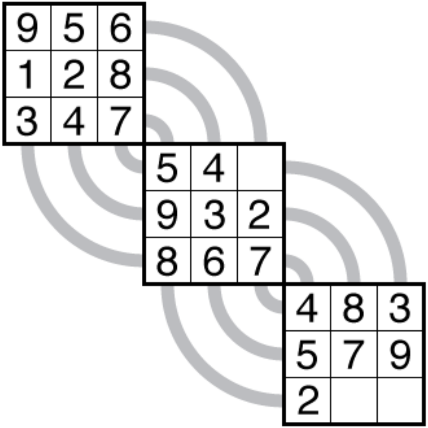

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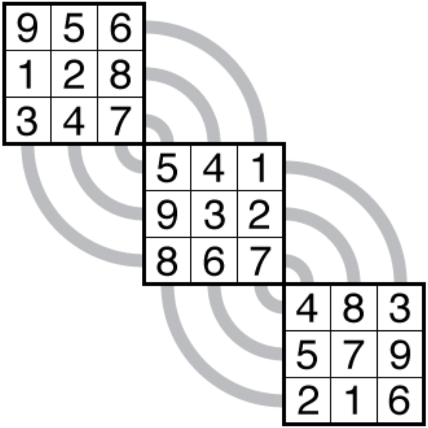

Cube Sudo-Kurve

We call this a cube Sudo-Kurve because we can unfold it into:

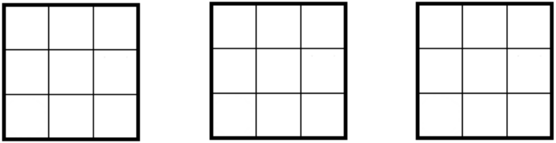

Figure: Empty Sudo-Cube grid

by flipping the middle square along its antidiagonal.

Number of Solutions

Theorem

The total number of valid fill-ins of the $3 \times 3 \times 3$ *Cube Sudo-Kurve is* $9! \times 40 = 14515200$.

The factor of 9! accounts for the fact that we can randomly permute the numbers in the first subgrid. Fixing that, we show that there are 40 ways to fill in the rest of the Sudo-Kurve.

Strategies

Observations:

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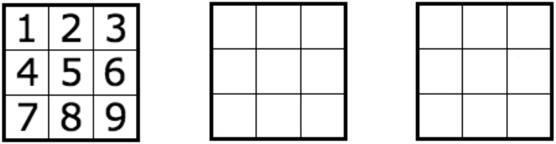
- If we know the locations of two instances of a symbol, we can figure out the third.

This enables us to already compute the case for the $3 \times 3 \times 3$ Cube Sudo-Kurve.

Counting

Once we fix the numbers in the first subgrid (e.g., to $1, 2, 3, 4, 5, 6, 7, 8, 9$),



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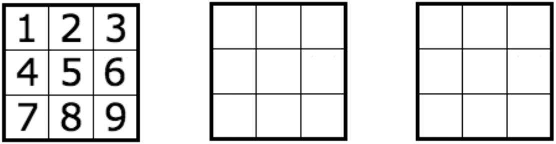

the first row of the second subgrid can be either

1. $4, 5, 6$ (numbers all from one row)
2. $4, 5, 7$ (numbers all from two rows and two columns)

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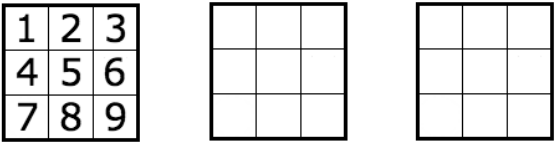


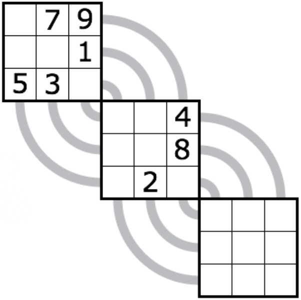
the first row of the second subgrid can be either

1. $4, 5, 6$ (numbers all from one row)
2. $4, 5, 7$ (numbers all from two rows and two columns)
3. $4, 5, 9$ (numbers all from two rows and three columns)

We can directly count in each of these cases, and we find the totals to be $16, 12, 12$ respectively, for 40 total.

Minimum Number of Clues

The minimum number of clues is at least 8. In fact, 8 is the minimum of clues:



Estimates for Higher Dimensions

An upper bound on an $n \times n \times n$ Sudo-Kurve is

$$\frac{((n^2)!)^n (n!)^{n^2(n-1)} (n-1)^{n^3(n-1)}}{n^{n^3(n-1)}} \approx \frac{((n^2)!)^n (n!)^{n^2(n-1)}}{e^{n^2(n-1)}}$$

which we get by considering the number of ways to assign coordinates to numbers.

Estimate Derivation

First, consider permutations of the multiset

$$\{\underbrace{1, 1, \ldots, 1}_{n \text{ times}}, \underbrace{2, 2, \ldots, 2}_{n \text{ times}}, \ldots, \underbrace{n, n, \ldots, n}_{n \text{ times}}\}$$

Examples

- Two permutations are **diverse** if no number is paired up with itself

$$\{1, 1, 1, 2, 2, 2, 3, 3, 3\}$$

$$\{2, 3, 2, 1, 3, 3, 1, 1, 2\}$$

Examples

- Two permutations are **complementary** if between the two permutations, all n^2 pairs of numbers are formed

$$\{1, 1, 1, 2, 2, 2, 3, 3, 3\}$$

$$\{1, 2, 3, 2, 3, 1, 3, 2, 1\}$$

Expressions

Let p_n be the number of permutations, d_n be the number of permutations diverse to a particular one, and c_n be the number of permutation complementary to a particular one. We have

- $$p_n = \binom{n^2}{\underbrace{n, n, \ldots, n}_{n \text{ times}}} = \frac{(n^2)!}{(n!)^n}.$$
- $$d_n = (n!)^n$$
- $$c_n = (-1)^n \int_0^\infty (L_n(x))^n e^{-x}\, dx < \frac{((n^2)!)^n (n!)^{n^2(n-1)} (n-1)^{n^3(n-1)}}{n^{n^3(n-1)}}$$

First Estimate

Our estimate is

$$p_n^{2n} \left(\frac{d_n}{p_n}\right)^{n(n-1)} \left(\frac{c_n}{p_n}\right)^{n}$$

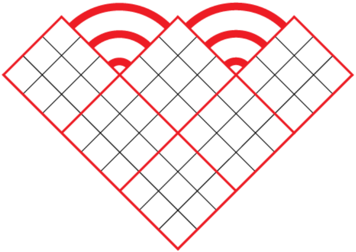
Generalization

That estimate easily generalizes to

$$p_n^{n(k-1)} \left(\frac{d_n}{p_n}\right)^{n(n-1)(k-1)/2} \left(\frac{c_n}{p_n}\right)^{nk(k-1)/2}$$

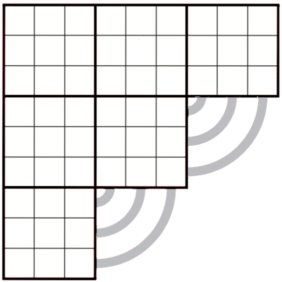
Other Grids

There are many variants. For instance, here is a heart-shaped Sudoku:



Heart Sudo-Kurve

There are 9! ways to fill in the top left-most subgrid.



Conjecture

There are 577338624 *ways to fill in this Sudo-Kurve.*

Future Work

Prove those above conjectures, and a few more. There are many, many more different possible shapes of Sudo-Kurves. There are also many more puzzles to consider.

Acknowledgements

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- Yu Zhao
- MIT PRIMES
- MIT Math Department
- My parents

References

GMPuzzles

Even, S., Gillis, J. (1976). Derangements and Laguerre polynomials. Mathematical Proceedings of the Cambridge Philosophical Society, 79(1), 135-143. doi:10.1017/S0305004100052154